A New Class of the Failure Quantile Models with Increasing, Decreasing and Bathtub-Shaped Failure Rates

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Abstract

The article considers a new fl exible distribution with different shape of quan- tile hazard rate. A new class developed is the sum of quantile functions of the generalized extreme value and weibull distributions. A new model has the probability density function whose sensitive skewness is a general case of probability density quantile function of the exponential distribution, probability density quantile function of the rayleigh distribution, exponential distribution and rayleigh distribution. Different properties and reliability characterization of this new model have been discussed and inference of parameters using percentile methods and L-moments methods are studied. To clarify the procedure of methods of estimation, two real data sets were performed.

Keywords: Generalized extreme distribution; weibull distribution; failure quantile function; L-moments quantile density function; quantile function.

1 Introduction

The distributions with four or more parameters don' t impose very strong restrictions on data. This is well illustrated by their ability to produce bathtub curves and this the ability to interpret the data appropriately and strong and the consequent reduction of the estimation errors and the cost in the ideal preventive maintenance problem, as opposed to distributions with one or two parameters.

In general, the failure rate and length of life of a component are an important problem in reliability analysis, in analyzing mechanical and electronic systems one

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uses the exponential distribution for systems or components with a constant failure rate, another one uses Raleigh, linear failure rate distribution or generalized exponential distribution when the components have an increasing failure rate. The lifetime of a component is expected to exhibit decreasing failure rate when its behavior over time is characterized by 'work-hardening' then we can use Lomax distribution (cf. Lomax (1954), Nassar (1988) and Kus (2007)). We may be used the bathtub failure rate if the failure rate is high at the start of the system or component and decreases toward a constant level. After acertain time, the system enters the war-out phase and the failure rate begins to increase (cf. Haupt and Schabe (1992) and Muldholkar and Srivastava (1993)). In practice often one needs to consider bathtub shaped failure rate function for one component, increasing failure rate for other components and decreasing failure rate for third components. Even though this case has been introduced in a good deal of reliability engineering systems, few practical distributions possessing this property have appeared in the studies.

Additionally, the quantile function is a very effective method for statistical analysis in many applications. For a non-negative random variable X with distribution function $F(x)$ and F^{-1} is the inverse function of F function. The quantile function $Q(\theta)$ is defined by

$$Q(\theta) = F^{-1}(x) = \inf\{x : F(t) \geq \theta\}, 0 < \theta < 1.$$

This function has a set of unique properties that are not enveloped by the distribution function such as the sum of the two quantiles is still quantile function. The most interesting approach to this issue has been proposed by Bassett and Koenker (1982), Castillo and Hadi (1994) and Gilchrist (2000).

In some reliability problems, it is convenient to work with median or some other percentile functions. Because mean is affected by outliers whereas median and percentiles are not. Additionally, to calculate the mean life we need to wait until every unit has failed, but in median time function, we need to wait until half of the units failed only. For several years great effort has been devoted to the study of the applications of percentile function in reliability analysis, such as the percentile failure rate, percentile residual lifetime and past lifetime of the components of the parallel systems which introduced by Noughabi and Izadkhah (2016), Sunoj et al. (2017) has also found that big role of the percentile function in reliability information theory and the new classes of distributions defined by a quantile function with application in reliability theory are discussed in many papers such as Sankaran et al. (2015), Sankaran et al. (2016) and Sankaran and Kumar (2018). In Nair and Sankaran (2009) it was defined that the derivative of

$$q(\theta) = \frac{dQ(\theta)}{d\theta},$$

as quanitle density function.

The purpose of this paper is to achieve three goals. The first goal is to introduce and study new practical distribution with decreasing, increasing and bathtub failure

rate. The Second goal is to provide practical and more flexible distribution by introducing a new quantile function, which has many important features in the theory of reliability and distribution theory. The proposed quantile function is derived by taking the sum of quantile functions of generalized extreme value type I and Weibull distributions. Extreme value theory seems to have originated mainly from the needs of astronomers in utilizing or rejection outlying observations. The early papers by Gumbel (1935; 1941) and Fuller (1914) and Griffith (1920) on the subject were highly specialized both in fields of applications and in methods of mathematical analysis. Let X_1 has generalized extreme value type I $(GE(\mu,\beta))$ distribution with scale parameter μ and shape parameter $\beta > 0$, then its survival and quantile functions are respectively given as by

$$\overline{F}_1(x) = 1 - e^{-t(x)},$$

$$Q_1(\theta) = \mu - \beta \ln(-\ln\theta), \ 0 < \theta < 1, \tag{1}$$

where $t(x) = \exp(-(x-\mu)/\beta)$. The Weibull distribution plays a vital role in survival analysis. It developed to predict the time to failure of mechanical components, but it has found application also in microbial inactivation for a variety of lethal agents. Let X_2 has Weibull distribution $(W(\alpha,\lambda))$ with scale parameter λ and shape parameter α, then its survival and quantile functions are respectively given as

$$\overline{F}_2(x) = \exp(-(x/\lambda)^{\alpha}), \ \lambda, \alpha > 0. \tag{2}$$

Therefore,

$$Q_2(\theta) = \lambda(-\ln(1-\theta))^{1/\alpha}, \ 0 < \theta < 1. \tag{3}$$

The rest of the paper is organized into five sections: Section 2 we present the suggested model of distributions, its members and study its basic properties and characterizations. Section 3 provides various reliability characterization. In Section 4, the inference of the distribution's parameters of a new family using percentile methods and L-moments methods are studied. To clarify the procedure of methods of estimation, two real data sets were performed. Finally, Section 5 describes brief conclusions of the results of this work.

2 The New Model and Basic Properties and Characterizations

Let X_1 and X_2 be two non-negative random variables with distribution functions $F_1(.)$ and $F_2(.)$ with quantile functions $Q_1(.)$ and $Q_2(.)$ respectively. Then

$$Q(\theta) = Q_1(\theta) + Q_2(\theta) \tag{4}$$

is a quantile function with the density quantial function $f(Q(\theta))$. Besides, the derivative of $Q(\theta)$, expressed as $q(\theta)$ is known as the quantile density function.

Equations (1), (3) and (4) express a new quantile function as

$$Q(\theta) = \mu + \lambda \left(-\ln\left(1-\theta\right)\right)^{1/\alpha} - \beta \ln\left(-\ln\theta\right), \tag{5}$$

where $0 < \theta < 1$, $\lambda, \alpha, \beta \in \mathbb{R}^{+}$. Thus, the new quantile density function, expressed as (Generalized Extreme Value Weibull quantile function $(GEW\left(\mu, \lambda, \alpha, \beta\right))$) is provided as

$$q(\theta) = \frac{\lambda}{\alpha\left(1-\theta\right)}\left(-\ln\left(1-\theta\right)\right)^{(1-\alpha)/\alpha} - \frac{\beta}{\theta \ln\theta}, \tag{6}$$

The suggested distribution can be expressed in term of density function $f\left(x\right)$ as

$$f\left(x\right) = \frac{\alpha F\left(x\right)\overline{F}\left(x\right)\ln F\left(x\right)}{\lambda F\left(x\right)\left(\ln F\left(x\right)\right)\left(-\ln\overline{F}\left(x\right)\right)^{(1-\alpha)/\alpha} - \alpha\beta\overline{F}\left(x\right)}. \tag{7}$$

The proposed density function for different parameters is shown in Figure (1). For all values of the parameters, the density is concave in θ and it tends to zero as $x \to 1$.

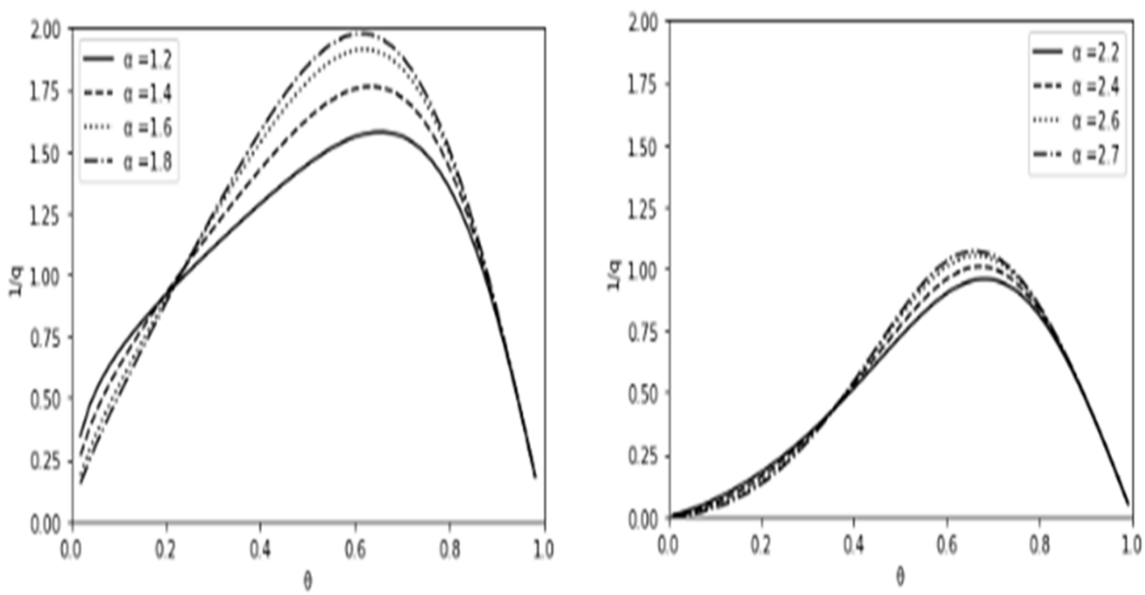

Figure (1): Plots of density function for different values of α.

This new distribution has a probability density quantile function with four parameters $\alpha, \lambda,\ \beta, \mu$. Therefore, the model can be considered more appropriate for practical and applied situations as well as being considered the general state of many probability distributions such as the probability density quantile function of the Exponential distribution (PDQE), the probability density quantile function of the Rayleigh distribution (PDQE), Exponential distribution (ED) and Rayleigh distribution (RD). They are described by the following cases in table (1):

Table (1): Sub-models of $GEW\left(\mu,\lambda,\alpha,\beta\right)$ models.

Reduced model	α	β	μ	λ	$q(\theta)$	$Q(\theta)$
PDQE	1	$\to 0$	$\to 0$	—	$\lambda/\left(1-\theta\right)$	$\lambda\left(-\ln\left(1-\theta\right)\right)$
ED	1	$\to 0$	$\to 0$	1	$\left(1-\theta\right)^{-1}$	$-\ln\left(1-\theta\right)$
PDQR	2	$\to 0$	$\to 0$	—	$\frac{\lambda(-\ln(1-\theta))^{-1/2}}{2(1-\theta)}$	$\lambda\left(-\ln\left(1-\theta\right)\right)^{1/2}.$
RD	2	$\to 0$	$\to 0$	1	$\frac{\left(-\ln(1-\theta)^{-1/2}\right)}{(2(1-\theta))}$	$\left(-\ln\left(1-\theta\right)\right)^{1/2}$

In the following results, we provide the random variable associated with the proposed quantile function (7).

Theorem 1 : If U_1 and U_2 are two random variables with quantile functions $Q_{U_1}(\theta)$ and $Q_{U_2}(\theta)$ and the distribution functions $G_{U_1}(\theta)$ and $G_{U_2}(\theta)$ respectively, let $U_1 \in GE\left(\mu,\beta\right)$ and $U_2 \in W\left(\alpha,\lambda\right)$ then we have

(i) the random variable $U_1+\lambda\left(-\ln\left(1-e^{-t(x)}\right)\right)^{1/\alpha}$ has $GEW\left(\mu,\lambda,\alpha,\beta\right).$

(ii) the random variable $U_2+\mu-\alpha\beta\left[\ln U_2-\ln\lambda\right]$ has $GEW\left(\mu,\lambda,\alpha,\beta\right).$

Proof: Sankaran et al. (2016) have demonstrated that the random variables

$$U_1+Q_{U_2}(G_{U_1}(\theta)),$$

and

$$U_2+Q_{U_1}(G_{U_2}(\theta)),$$

are random variables corresponds to the quantile function $\Phi\left(\theta\right)=Q_{U_1}(\theta)+Q_{U_2}(\theta)$. If $U_1 \in GE\left(\mu,\beta\right)$ and $U_2 \in W\left(\alpha,\lambda\right)$, by using (1) and (2), then we have

$$U_1+Q_{U_2}(G_{U_1}(\theta))=U_1+\lambda\left(-\ln\left(1-e^{-t(U_1)}\right)\right)^{1/\alpha},$$

by (4), we can conclude that $U_1+Q_{U_2}(G_{U_1}(\theta)\in GEW\left(\mu,\lambda,\alpha,\beta\right).$ The proof (ii) is similar to that of (i).

2.1 Quartile coefficients

We can characterization the derivative of $f(x)$ by the distribution function as follows

$$\begin{aligned} f^{\backprime}\left(x\right) &= \frac{\alpha f\left(x\right)\left[\lambda\varphi\left(x\right)\psi\left(x\right)-\alpha\beta\overline{F}\left(x\right)\right]\left[\ \ln F\left(x\right)-2\varphi\left(x\right)+\overline{F}\left(x\right)\right]}{\left[\lambda\varphi\left(x\right)\psi\left(x\right)-\alpha\beta\overline{F}\left(x\right)\right]^2} \\ &\quad -\frac{\alpha f\left(x\right)\overline{F}\left(x\right)\varphi\left(x\right)\psi\left(x\right)\left[\lambda\left(\ln F\left(x\right)+1-\left(\frac{1-\alpha}{\alpha}\right)\frac{\varphi(x)}{\overline{F}(x)\ln\overline{F}(x)}\right)+\alpha\beta\right]}{\left[\lambda\varphi\left(x\right)\psi\left(x\right)-\alpha\beta\overline{F}\left(x\right)\right]^2} \end{aligned}$$

where $\varphi(x) = F(x)\ln F(x)$, which is called lack information in information theory and $\psi(x) = \left(-\ln \overline{F}(x)\right)^{(1-\alpha)/\alpha}$.

Now we can derive Quartile coefficients of location (median (M)), dispersion (the interquartile range (IQ)), skewness (Galton's coefficient (G)) and kurtosis (Moor's coefficient (MK)) respectively as

$$M = Q(\frac{1}{2}) = \mu + \lambda(\ln(2))^{1/\alpha} - \beta\ln(\ln 2),$$

$$\begin{aligned} IQ &= Q(\frac{3}{4}) - Q(\frac{1}{4}) \\ &= 1.57\beta - \lambda(0.288)^{1/\alpha} + \lambda(1.39)^{1/\alpha}, \end{aligned}$$

$$\begin{aligned} G &= \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{IQ} \\ &= \frac{0.92\beta + \lambda(0.29)^{1/\alpha} + \lambda(1.39)^{1/\alpha} + 2\mu - 2\left(\mu + \lambda(\ln(2))^{1/\alpha} - \beta\ln(\ln 2)\right)}{1.57\beta - \lambda(0.29)^{1/\alpha} + \lambda(1.39)^{1/\alpha}}, \end{aligned}$$

and

$$\begin{aligned} MK &= \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{IQ} \\ &= \frac{\eta(\alpha,\lambda,\beta)}{1.57\beta - \lambda(0.29)^{1/\alpha} + \lambda(1.39)^{1/\alpha}}, \end{aligned}$$

where $\eta(\alpha,\lambda,\beta) = -\lambda\ln\left(\frac{8}{7}\right)^{1/\alpha} + \lambda\ln\left(\frac{8}{5}\right)^{1/\alpha} - \lambda\ln\left(\frac{8}{3}\right)^{1/\alpha} + \lambda\ln(8)^{1/\alpha} - \beta\ln\ln\left(\frac{8}{7}\right) + \beta\ln\ln\left(\frac{8}{5}\right) - \beta\ln\ln\left(\frac{8}{3}\right) + \beta\ln\ln(8)$.

2.2 L-moments

The moments are useful tools for analyzing important properties of T when they exist. However, moments functions may not exist. Even when it exists or it may have some practical shortcomings, especially in situations where the data are censored, or when the underlying distribution is skewed or heavy- tailed. In such cases, the L-moments to be considered next can provide a competing alternative to the conventional moments.

The rth L moment is given by,

$$L_r = \int_0^1 \sum_{s=0}^{r-1} (-1)^{r-1-s} \binom{r-1}{s} \binom{r-1+s}{s} \theta^s Q(\theta)\, d\theta. \tag{8}$$

Since

$$\begin{aligned}&\sum_{s=0}^{r-1}(-1)^{r-1-s}\binom{r-1}{s}\binom{r-1+s}{s}\theta^{s}Q(\theta) \qquad (9)\\ =\ & -(-1)^{r}\ {}_2F_1\left[r,1-r;1;\theta\right]\left(\mu+\lambda\left(-\ln\left(1-\theta\right)\right)^{1/\alpha}-\beta\ln\left(-\ln\theta\right)\right)\end{aligned}$$

Hence, by using (5), (8) and (9), we obtain the first four L moments as follows:

$$L_1=\gamma\beta+\mu+\lambda\Gamma\left(1+\frac{1}{\alpha}\right),$$

$$L_2=\left(\lambda-2^{-1/\alpha}\lambda\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\beta\ln\left(2\right),$$

$$L_3=\lambda\left(1-3\left(2^{-1/\alpha}\right)+2\left(3^{-1/\alpha}\right)\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\beta\ln\left(\frac{9}{8}\right),$$

and

$$L_4=\lambda\left(1-3\left(2^{(\alpha-1)/\alpha}\right)+10\left(3^{-1/\alpha}\right)-5\left(4^{-1/\alpha}\right)\right)\Gamma\left(1+\frac{1}{\alpha}\right)+2\beta\ln\left(\frac{256}{243}\right),$$

for all $Re\left[\frac{1}{\alpha}\right]>-1$ and γ is Euler Gamma, $\Gamma\left(.\right)$ is gamma function and $Re\left[\frac{1}{\alpha}\right]$ gives the real part of the complex number $\frac{1}{\alpha}$.

For the model (7), L-coefficient of variation (ϕ_1), L-coefficient of skewness (ϕ_2) and coefficient of kurtosis (ϕ_3) have the following expressions:

$$\phi_1=\frac{L_2}{L_1}=\frac{\left(\lambda-2^{-1/\alpha}\lambda\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\beta\ln\left(2\right)}{\gamma\beta+\mu+\lambda\Gamma\left(1+\frac{1}{\alpha}\right)},$$

$$\phi_2=\frac{L_3}{L_2}=\frac{\lambda\left(1-3\left(2^{-1/\alpha}\right)+2\left(3^{-1/\alpha}\right)\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\beta\ln\left(\frac{9}{8}\right)}{\left(\lambda-2^{-1/\alpha}\lambda\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\beta\ln\left(2\right)},$$

$$\phi_3=\frac{L_4}{L_2}=\frac{\lambda\left(1-3\left(2^{(\alpha-1)/\alpha}\right)+10\left(3^{-1/\alpha}\right)-5\left(4^{-1/\alpha}\right)\right)\Gamma\left(1+\frac{1}{\alpha}\right)+2\beta\ln\left(\frac{256}{243}\right)}{\left(\lambda-2^{-1/\alpha}\lambda\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\beta\ln\left(2\right)}.$$

2.3 Order statistics

In this subsection, we derive an explicit expression for the density of the ith order statistic $X_{(i:n)}$, say $f_{(i:n)}(x)$, in a random sample of size n from $GEW\left(\mu,\lambda,\alpha,\beta\right)$ distribution and the probability density functions of the smallest, largest and median order statistic. Besides, the joint distribution of ith and j^{th} as well as r^{th} moment about zero of the ith order statistics are investigated.

Let $T_1, T_2, ..., T_n$ be a random sample having $GEW\left(\mu, \lambda, \alpha, \beta\right)$ distribution and let $T_{(1)}, T_{(2)}, ..., T_{(n)}$ be the corresponding order statistics. The probability density function of ith order statistic is given by:

$$f_{(i:n)}(t) = \frac{1}{B(i, n-i+1)} \left[F_T(t)\right]^{i-1} \left[1 - F_T(t)\right]^{n-i} f_T(t), \quad 1 \leq i \leq n \tag{10}$$

Using the binomial series expansion of $\left[1 - F_T(t)\right]^{n-i}$, we obtain

$$\left[1 - F_T(t)\right]^{n-i} = \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[F_T(t)\right]^j. \tag{11}$$

Inserting (7) and (11) into (10), we have

$$f_{(i:n)}(t) = \sum_{j=0}^{n-i} \frac{(-1)^j \alpha \binom{n-i}{j} \left[F_T(t)\right]^{j+i} \overline{F}_T\left(t\right) \ln F_T\left(t\right)}{B(i, n-i+1) \left(\lambda F_T\left(t\right) \left(\ln F_T\left(t\right)\right) \left(-\ln \overline{F}_T\left(t\right)\right)^{(1-\alpha)/\alpha} - \alpha\beta \overline{F}_T\left(t\right)\right)},$$

where $1 \leq i \leq n$. Therefore,

$$E\left[T_{(i:n)}\right] = \int_0^\infty \sum_{j=0}^{n-i} \frac{(-1)^j \alpha \binom{n-i}{j} t \left[F_T(t)\right]^{j+i} \overline{F}_T\left(t\right) \ln F_T\left(t\right)}{B(i, n-i+1) \left(\lambda F_T\left(t\right) \left(\ln F_T\left(t\right)\right) \left(-\ln \overline{F}_T\left(t\right)\right)^{(1-\alpha)/\alpha} - \alpha\beta \overline{F}_T\left(t\right)\right)} dt$$

In this way, we obtain $E\left[T_{(i:n)}\right]$ by using quantile function as

$$E\left[T_{(i:n)}\right] = \int_0^1 \sum_{j=0}^{n-i} Q\left(\theta\right) \frac{(-1)^{j-((1-\alpha)/\alpha)} \alpha \binom{n-i}{j} t \theta^{j+i} \overline{\theta} \ln \theta}{B(i, n-i+1) \left(\lambda \theta \ln \theta \left(\ln \overline{\theta}\right)^{(1-\alpha)/\alpha} - \alpha\beta\overline{\theta}\right)} d\theta,$$

where $\overline{\theta} = 1 - \theta$. For the family of distributions (5), the first order statistic $T_{(1:n)}$ has the quantile function,

$$\begin{aligned} Q_{(1:n)}(\theta) &= Q\left(1 - \overline{\theta}^{\frac{1}{n}}\right) \\ &= \mu + \lambda \left(-\ln \overline{\theta}^{\frac{1}{n}}\right)^{1/\alpha} - \beta \ln\left(-\ln\left(1 - \overline{\theta}^{\frac{1}{n}}\right)\right) \end{aligned}$$

and the last order statistic $T_{(n:n)}$ has the quantile function,

$$\begin{aligned} Q_{(n:n)}(\theta) &= Q\left(\theta^{\frac{1}{n}}\right) \\ &= \mu + \lambda \left(-\ln\left(1 - \theta^{\frac{1}{n}}\right)\right)^{1/\alpha} - \beta \ln\left(-\ln \theta^{\frac{1}{n}}\right). \end{aligned}$$

In addition, the joint distribution of the ith and j^{th} order statistics $f_{(i:j:n)}(t_i, t_j),\ i < j$ can be expressed as:

$$\begin{aligned} f_{(i:j:n)}(t_i,t_j) &= C_{(i:j:n)}\left[F_T(t_i)\right]^{i-1}\left[1-F_T(t_j)\right]^{n-j}\left[F_T(t_j)-F_T(t_i)\right]^{j-i-1} f_T(t_i)f_T(t_j) \\ &= C_{(i:j:n)}\left[F_T(t_i)\right]^{i-1}\sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\left[F_T(t_j)\right]^s\left[F_T(t_j)-F_T(t_i)\right]^{j-i-1} \\ &\quad\times\frac{\alpha^2 F_T(t_i)\,\overline{F}_T(t_i)\ln F_T(t_i)}{\lambda F_T(t_i)\left(\ln F_T(t_i)\right)\left(-\ln\overline{F}_T(t_i)\right)^{(1-\alpha)/\alpha}-\alpha\beta\overline{F}_T(t_i)} \\ &\quad\times\frac{F_T(t_j)\,\overline{F}_T(t_j)\ln F_T(t_j)}{\lambda F_T(t_j)\left(\ln F_T(t_j)\right)\left(-\ln\overline{F}_T(t_j)\right)^{(1-\alpha)/\alpha}-\alpha\beta\overline{F}_T(t_j)} \end{aligned}$$

for $-\infty < t_i < t_j < \infty$ and $C_{(i:j:n)} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$. Then, the minimum and maximum joint probability density of $CIR\ (\alpha, \beta)$, denoted by $f_{(1:n:n)}(t_1, t_n)$, can be obtained from (10) substituting $i = 1$ and $j = n$, as follows

$$\begin{aligned} f_{(1:n:n)}(t_1,t_n) &= C_{(1:n:n)}\left[F_T(t_n)-F_T(t_1)\right]^{n-2} \\ &\quad\times\frac{\alpha^2 F_T(t_1)\,\overline{F}_T(t_1)\ln F_T(t_1)}{\lambda F_T(t_1)\left(\ln F_T(t_1)\right)\left(-\ln\overline{F}_T(t_1)\right)^{(1-\alpha)/\alpha}-\alpha\beta\overline{F}_T(t_1)} \\ &\quad\times\frac{F_T(t_n)\,\overline{F}_T(t_n)\ln F_T(t_n)}{\lambda F_T(t_n)\left(\ln F_T(t_n)\right)\left(-\ln\overline{F}_T(t_n)\right)^{(1-\alpha)/\alpha}-\alpha\beta\overline{F}_T(t_n)}. \end{aligned}$$

3 Reliability Properties

3.1 The hazard quantile function

A basic quantity, foundational in survival analysis, is the hazard quantile function. This function is also known as the conditional failure rate in reliability, the force of mortality in demography, the age-specific failure rate in epidemiology, the inverse of the Mill's ratio in economics or simply as the failure rate. The hazard quantile function $H(\theta)$ can be represented by

$$H(\theta) = \frac{f(Q(\theta))}{1-f(Q(\theta))} = \left((1-\theta)\,q(\theta)\right)^{-1} = \frac{\alpha\theta\ln\theta}{\lambda\theta\left(-\ln\overline{\theta}\right)^{(1-\alpha)/\alpha}\ln\theta-\alpha\beta\overline{\theta}} \tag{12}$$

where $\overline{\theta} = 1-\theta$. Inserting (1) and (2) in (12) we get a characterization of hazard quantile by $Q_1(\theta)$ and $Q_2(\theta)$ as follows

$$H(\theta) = \frac{\alpha\theta\exp\left(\left(\mu-Q_1(\theta)\right)/\beta\right)}{\theta\lambda^{\alpha}Q_2(\theta)^{1-\alpha}\exp\left(\left(\mu-Q_1(\theta)\right)/\beta\right)+\alpha\beta\overline{\theta}}.$$

Survival analysis is focused on the concept of quantile failure rate in many applications such as reliability theory and medical context.

3.2 Some typical bathtub curves

The new class distributions include the main standard hazard shapes (increasing, decreasing and upside-down bathtub) for different choices of parameters, which we believe is an important property for any parametric family, see Figures (4) and (5).

The shape of the quantile failure rate function is determined by the derivative of $H(\theta)$, which is provided as,

$$H^{\backprime}(\theta) = \frac{\overline{\theta}^{-1}\left(\alpha^2\beta\ln\overline{\theta}^2\left[2\theta-\theta^2-\overline{\theta}\ln\theta-1\right]+\lambda\theta^2\ln\theta^2\left(-\ln\overline{\theta}\right)^{1/\alpha}(\alpha-1)\right)}{\left(\alpha\beta\theta\ln\overline{\theta}-\alpha\beta\ln\overline{\theta}-\theta\lambda\left(-\ln\overline{\theta}\right)^{1/\alpha}\ln\theta\right)^2},$$

We found $\left(\alpha\beta\theta\ln\overline{\theta}-\alpha\beta\ln\overline{\theta}-\theta\lambda\left(-\ln\overline{\theta}\right)^{1/\alpha}\ln\theta\right)^2 > 0$, then the kind of quantile failure rate depend on sign of the following formula

$$\pi(\alpha,\lambda,\beta,\theta) = \overline{\theta}^{\,-1}\left(\alpha^2\beta\ln\overline{\theta}^{\,2}\left[2\theta-\theta^2-\overline{\theta}\ln\theta-1\right]+\lambda\theta^2\ln\theta^2\left(-\ln\overline{\theta}\right)^{1/\alpha}(\alpha-1)\right). \tag{13}$$

The failure rate reaches its minimum if we can minimize the $\pi(\alpha,\lambda,\beta,\theta)$ with respect to θ. This minimize of θ is obtained as a solution of the following fixed-point type equation:

$$g(\theta) = \theta, \tag{14}$$

where

$$g(\theta) = \sqrt{\frac{\alpha^2\beta\ln\overline{\theta}^{\,2}\left[\theta^2+\overline{\theta}\ln\theta-2\theta+1\right]}{\lambda\ln\theta^2\left(-\ln\overline{\theta}\right)^{1/\alpha}(\alpha-1)}}.$$

The solution of (14) can be obtained by a simple iterative procedure. Suppose we start with an initial guess θ_0, then the next iteration θ_1 can be obtained as $\theta_1 = g(\theta_0)$, similarly, $\theta_2 = g(\theta_1)$ and so on. Finally the iterative procedure should be stopped when $[\theta_i - \theta_{i+1}] < \varepsilon$, where ε is a preassigned tolerance value. If burn-in is to be applied to increase field reliability, the burn-in time should not be longer than to θ_i. In addition, after to θ_i, replacement should be considered because the failure rate will start increasing and at certain times, it may be too high.

It has been found that the hazard quantile function $H(\theta)$ has constant if $\lambda = 1$, $\theta = 0.5$, $\alpha \in [0,1[$, $\beta > 0$ and

$$\pi(\alpha,\lambda,\beta,\theta) = \delta\beta,$$

and when

$$\beta > \left(\lambda\theta(1-\alpha)\left(-\ln\overline{\theta}\right)^{1/\alpha}\ln\theta^2\right)/\left(2\alpha^2\ln\overline{\theta}^2\right).$$

Therefore $H(\theta)$ has an increasing hazard quantile model. It is evident that $H(\theta)$ belong to decreasing hazard quantile model if

$$\beta < \left(\lambda\theta(1-\alpha)\left(-\ln\overline{\theta}\right)^{1/\alpha}\ln\theta^2\right)/\left(2\alpha^2\ln\overline{\theta}^2\right).$$

Then, the new class distributions include the main standard hazard shapes (constant, increasing, decreasing and upside-down bathtub) for different choices of parameters, which we believe is an important property for any parametric family. Some plots using (7) can be found in Figure (2).

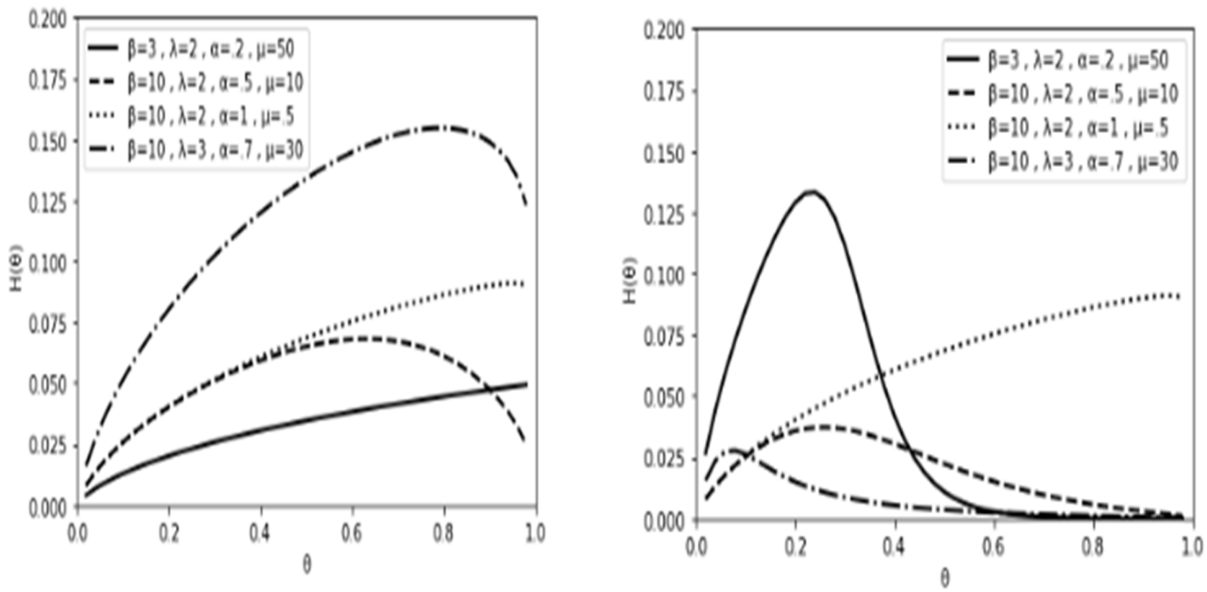

Figure (2): Some typical bathtub-shaped hazard percentile rate functions using different values of the parameters.

3.3 Some basic reliability functions

Let T be a non-negative random variable denote the lifetime of a unit or organism, having absolutely continuous distribution function F. In many reliability problems, there are interesting to consider variables of the following kind

$$T_t = [T - t \mid T > t], t \in \{x; x \ni F_T(x) < 1\},$$

and having distribution function $F_{T_t}(s) = P[T - t \leq s \mid T \geq t]$ and which is known in the literature as residual time. For any life variable $T \geq 0$, the residual life variable $T_t = [T - t \mid T \geq t]$, where $t \in (0, l_T)$ and $l_T = \sup\{t \ni F_T(t) < 1\}$, is a nonnegative random variable representing the remaining life of T at age t. Hence, if $F(.)$ is the distribution function of T and $\overline{F}(.) = 1 - F(.)$ is its survival function, thus the survival function of T_t, is given by

$$\overline{F}_{T_t}(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \qquad x \geq 0, \quad t \geq 0.$$

The mean residual life at time t can be represented as

$$\mu_T(t) = E[T - t \mid T \geq t] = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)}, \quad t \geq 0.$$

Now, the reversed hazard quantile and mean residual quantile functions respectively provided as

$$\delta_1(\theta) = (\theta q(\theta))^{-1} = \frac{\alpha\theta\overline{\theta}\ln\theta}{\theta\left[\theta\ln\theta\lambda\left(-\ln\overline{\theta}\right)^{(1-\alpha)/\alpha} - \alpha\overline{\theta}\beta\right]}$$

and

$$\begin{aligned}\delta_2(\theta) &= \theta^{-1}\int_0^\theta (xq(x))^{-1}dx \\ &= (\alpha\theta)^{-1}\left(\lambda\left(-\Gamma\left(\frac{1}{\alpha}\right) + \Gamma\left(\frac{1}{\alpha}, -\ln\overline{\theta}\right) + \alpha\left(-\ln\overline{\theta}\right)^{1/\alpha}\right) - \alpha\beta\ li(\theta)\right).\end{aligned}$$

To calculate the mean residual life of a system we need to wait until every unit of the system failed, but if we use the quantile residual life function we only need to wait until quantile (half) units have failed. Besides, in some instances, the mean residual life may not even exist. As a result of some reliability problems, it is convenient to work with the median or some other quantile of the residual life rather than the mean residual life when there are heavy-tailed distributions.

If θ is some number between 0 and 1, the quantile residual function of T, denoted by $\delta_{T,\theta}(t)$ is defined for any $t \geq l_T$ by letting $\delta_{T,\theta}(t)$ be the quantile residual life function of T_t. That is,

$$\delta_{T,\theta}(t) = \inf\left[x : F_{T_t} \geq \theta\right] = F^{-1}\left[1 - \overline{\theta}\,\overline{F}(t)\right] - t.$$

Nair and Sankaran (2009) have defined the quantile residual life in terms of quantile functions as follows

$$\begin{aligned}\delta_{T,\theta}(t) &= (1-\theta)^{-1}\int_\theta^1 [Q(x) - Q(\theta)]\,dx, \\ &= (1-\theta)^{-1}\int_\theta^1 (1-x)\,q(x)dx, \qquad t \geq l_T;\end{aligned}$$

where $\overline{\theta} = 1 - \theta$. It follows from (6) that $\delta_{T,\theta}(t)$ of the new class can be provided as

$$\delta_{T,\theta}(t) = \overline{\theta}^{-1}\left[\lambda\Gamma\left(\frac{1+\alpha}{\alpha}, -\ln\overline{\theta}\right) + \lambda(\theta-1)\left(-\ln\overline{\theta}\right)^{1/\alpha} + \beta(\gamma + \ln(-\ln\theta)) - \beta\ li(\theta)\right], \tag{15}$$

where $\Gamma(.,.)$ is upper incomplete gamma, γ is Euler gamma and $li(x)$ is logarithmic integral function of x.

As reported by Nair and Vineshkumar (2010), the variance quantile residual function can be calculated by

$$\sigma_{T,\theta}(t) = (1-\theta)^{-1} \int_{\theta}^{1} Q^2(x)dx - (\delta_{T,\theta}(t) + Q(\theta))^2 .$$

From (6) and (15) imply that

$$\sigma_{T,\theta}(t) = (1-\theta)^{-1} \left[\lambda^2 \Gamma\left(\frac{2+\alpha}{\alpha}, -\ln \overline{\theta} \right) + \phi(\alpha,\beta,\lambda) \right] - (\delta_{T,\theta}(t) + Q(\theta))^2 .$$

where

$$Q^2(\theta) = \lambda^2 (-\ln(1-\theta))^{2/\alpha} - 2\lambda\beta (-\ln(1-\theta))^{1/\alpha} \ln(-\ln\theta) + \beta^2 \ln(-\ln\theta)^2 , \quad (16)$$

$$\int_{\theta}^{1} \lambda^2 (-\ln(1-x))^{2/\alpha} dx = \lambda^2 \Gamma\left(\frac{2+\alpha}{\alpha}, -\ln\overline{\theta} \right),$$

and

$$\phi(\alpha,\beta,\lambda) = \int_{\theta}^{1} \beta^2 \ln(-\ln x)^2 dx - \int_{\theta}^{1} 2\lambda\beta (-\ln(1-x))^{1/\alpha} \ln(-\ln x) dx. \quad (17)$$

Using Newton-Cotes method we find the solution of (17).

However, it is reasonable to presume that in many realistic situations, the random life variable is not necessarily related to the future but can also refer to the past. For instance, consider a system whose state is observed only at certain preassigned inspection times. If at time t the system is inspected for the first time and it is found to be "down," then the failure relies on the past. It thus seems natural to study a notion which is dual to the residual life, in the sense which it refers to past time and not to future that is knew as the inactivity time or reversed residual life. The mean inactivity lifetime function, $E[T_t]$, as to be presented as

$$E\left[T_{(t)}\right] = E[t - T \mid T \le t] = \frac{\int_0^t F(u)du}{F(t)}.$$

In addition, the quantile reversed residual function of T ($\overline{\delta}_{T,\theta}(t)$) can be defined as

$$\overline{\delta}_{T,\theta}(t) = t - \overline{F}^{\,-1}\left[\theta + \overline{\theta}\,\overline{F}(t)\right], \quad t \ge l_T.$$

Nair and Sankaran (2009) have represented the quantile reversed residual life in terms of quantile function as follows

$$\overline{\delta}_{T,\theta}(t) = \theta^{-1} \int_0^{\theta} [Q(\theta) - Q(x)]\, dx, \tag{18}$$

It follows from (6) and (18) that $\overline{\delta}_{T,\theta}(t)$ of the new class can be provided as

$$\overline{\delta}_{T,\theta}(t) = \theta^{-1} \left[\lambda\Gamma\left(\frac{1+\alpha}{\alpha}, -\ln\overline{\theta}\right) + \theta\lambda\left(-\ln\overline{\theta}\right)^{1/\alpha} - \beta\ li(\theta) - \lambda\Gamma\left(\frac{1+\alpha}{\alpha}\right)\right], \tag{19}$$

where $\Gamma(.)$ is a gamma function. As reported by Nair and Vineshkumar (2010), the variance quantile reversed residual function can be calculated by

$$\overline{\sigma}_{T,\theta}(t) = \theta^{-1} \int_0^{\theta} Q^2(x)dx - \left(Q(\theta) - \overline{\delta}_{T,\theta}(t)\right)^2.$$

From (6), (16) and (19) imply that

$$\begin{aligned}
\overline{\sigma}_{T,\theta}(t) &= \theta^{-1} \int_0^{\theta} Q^2(x)dx - \left(Q(\theta) - \overline{\delta}_{T,\theta}(t)\right)^2 \\
&= \theta^{-1} \left[\lambda^2 \left(\Gamma\left(\frac{2+\alpha}{\alpha}\right) - \Gamma\left(\frac{2+\alpha}{\alpha}, -\ln\overline{\theta}\right)\right) + \overline{\phi}(\alpha,\beta,\lambda)\right] - \left(Q(\theta) - \overline{\delta}_{T,\theta}(t)\right)^2.
\end{aligned}$$

$$\int_0^{\theta} \left(-\ln(1-x)\right)^{2/\alpha} dx = \Gamma\left(\frac{2+\alpha}{\alpha}\right) - \Gamma\left(\frac{2+\alpha}{\alpha}, -\ln\overline{\theta}\right),$$

and

$$\overline{\phi}(\alpha,\beta,\lambda) = \int_0^{\theta} \beta^2 \ln(-\ln x)^2\, dx - \int_0^{\theta} 2\lambda\beta\left(-\ln(1-x)\right)^{1/\alpha} \ln(-\ln x)\, dx. \tag{20}$$

Using Newton-Cotes method we find the solution of (20).

Let X be a non-negative random variable with distribution function F and with a finite mean. The generalized Lorenz curve $(\widetilde{L}_X)$, it is defined by the following transform:

$$\begin{aligned}
\widetilde{L}_X(\theta) &= \int_0^{\theta} F^{-1}(u)\, du, \qquad \theta \in [0,1]. \\
&= \lambda\Gamma\left(\frac{1+\alpha}{\alpha}\right) - \lambda\Gamma\left(\frac{1+\alpha}{\alpha}, -\ln\overline{\theta}\right) + \beta(\ li(\theta) - \theta\ln(-\ln\theta))
\end{aligned}$$

Applications, properties and interpretations of Lorenz curve in the statistical theory, reliability theory and in economics can be found in Ahmad and Kayid (2005, 2006, 2007), Li and Zuo (2004), and Li (2004) and among other.

In statistics, L-moments are a sequence of statistics used to summarize the shape of a probability distribution. The sth L-moment residual quantile function of X studied in Kupka and Loo (1989) which is given by

$$\psi_r(\theta) = \sum_{j=0}^{s} (-1)^j \binom{s-1}{j}^2 \int_{\theta}^{1} \left(\frac{x-\theta}{1-\theta}\right)^{s-j-1} \left(\frac{1-x}{1-\theta}\right)^j \frac{Q(x)}{1-\theta} dx. \tag{21}$$

In particular, from (21) we have

$$\begin{aligned}\psi_1(\theta) &= (1-\theta)^{-1}\int_\theta^1 Q(x)\,dx \\ &= (1-\theta)^{-1}\left(\gamma\beta+\lambda\Gamma\left(\frac{1+\alpha}{\alpha},-\ln\overline{\theta}\right)+\beta\theta\ln(-\ln\theta)+\beta\ li(\theta)\right)\end{aligned}$$

and

$$\begin{aligned}\psi_2(\theta) &= (1-\theta)^{-2}\int_\theta^1 (2x-\theta-1)\,Q(x)\,dx \\ &= (1-\theta)^{-2}\left(\beta(\ln 2-\theta\gamma)-2^{-1/\alpha}\left(2^{1/\alpha}\beta\,\mathrm{Ei}[2\ln\theta]+\lambda\Gamma\left(\frac{1+\alpha}{\alpha},-2\ln\overline{\theta}\right)\right)\right) \\ &\quad+(1-\theta)^{-2}\left(2^{1/\alpha}\left((\theta-1)\lambda\Gamma\left(\frac{1+\alpha}{\alpha},-\ln\overline{\theta}\right)+\beta(\theta\ln(-\ln\theta)-(1+\theta)\,li(\theta))\right)\right)\end{aligned}$$

where $\Gamma(.,.)$ is upper incomplete gamma, γ is Euler gamma, $li(x)$ is logarithmic integral function of x and $\mathrm{Ei}[x]$ is the exponential integral function.

The first and second L-moments of reversed residual quantile function for many distributions is studied in studies such as Sankaran et al. (2015) and Sankaran and Kumarm (2018) as

$$\begin{aligned}\overline{\psi}_1(\theta) &= \theta^{-1}\int_0^\theta Q(x)\,dx \\ &= \theta^{-1}\left(\lambda\Gamma\left(\frac{1+\alpha}{\alpha}\right)-\lambda\Gamma\left(\frac{1+\alpha}{\alpha},-\ln\overline{\theta}\right)+\beta(-\theta\ln(-\ln\theta)+li(\theta))\right)\end{aligned}$$

and

$$\begin{aligned}\overline{\psi}_2(\theta) &= \theta^{-2}\int_\theta^1 (2x-\theta-1)\,Q(x)\,dx \\ &= \theta^{-2}\left(\beta(\ln 2-\theta\gamma)-2^{-1/\alpha}\left(2^{1/\alpha}\beta\,\mathrm{Ei}[2\ln\theta]+\lambda\Gamma\left(\frac{1+\alpha}{\alpha},-2\ln\overline{\theta}\right)\right)\right) \\ &\quad+(1-\theta)^{-2}\left(2^{1/\alpha}\left((\theta-1)\lambda\Gamma\left(\frac{1+\alpha}{\alpha},-\ln\overline{\theta}\right)+\beta(\theta\ln(-\ln\theta)-(1+\theta)\,li(\theta))\right)\right)\end{aligned}$$

where $\Gamma(.,.)$ is upper incomplete gamma, γ is Euler gamma, $li(x)$ is logarithmic integral function of x and $\mathrm{Ei}[x]$ is the exponential integral function.

Gini's mean difference for the random variable T_t which defined in (13) is expressed in terms of the quantile function as

$$\begin{aligned}\mathbb{G}(Q(\theta)) &= 2\int_\theta^1 \frac{(1-x)(x-\theta)}{(1-\theta)^2}q(x)\,dx, \\ &= 2\beta(-\gamma\theta+i\pi\theta-\mathrm{Ei}[2\ln\theta]+\ln 2-\theta\ln\ln\theta+\ li(\theta)+\theta\ li(\theta)) \\ &\quad-\alpha^{-1}2^{\frac{\alpha-1}{\alpha}}\lambda\Gamma\left(\tfrac{1}{\alpha},-2\ln\overline{\theta}\right)-2\lambda\alpha^{-1}\Gamma\left(\tfrac{1}{\alpha},-\ln\overline{\theta}\right)(\theta-1).\end{aligned}$$

Moreover, $\mathbb{G}\left(Q\left(\theta\right)\right)/2$ is half the mean difference of T, which is used as measure of dispersion in theoretical and applied economic studies such as poverty and income.

The score function is a widely accepted statistical method, which has many applications in statistical data modeling and reliability context, where there is a relationship between it and hazard quantile function (the equivalent of the failure rate function) (see, Nair et al. (2012) and Parzen (1979)). The score function has the form

$$\mathbb{S}\left(\theta\right)=\frac{q^{`}\left(\theta\right)}{q^{2}\left(\theta\right)},$$

where $q^{`}\left(x\right)=\partial q\left(x\right)/\partial x$. For the modelling of distributions (6), $\mathbb{S}\left(\theta\right)$ is defined as

$$\mathbb{S}\left(\theta\right)=\frac{\pi_{1}\left(\alpha,\lambda,\beta;\theta\right)}{\pi_{2}\left(\alpha,\lambda,\beta;\theta\right)},$$

where

$$\pi_{1}\left(\alpha,\lambda,\beta;\theta\right)=\frac{\lambda\left(\frac{1-\alpha}{\alpha}\right)\left(-\ln\overline{\theta}\right)^{(1-2\alpha)/\alpha}}{\alpha\overline{\theta}^{2}}+\frac{\lambda\left(-\ln\overline{\theta}\right)^{(1-\alpha)/\alpha}}{\alpha\overline{\theta}^{2}}+\frac{\beta}{\theta^{2}\ln\theta^{2}}+\frac{\beta}{\theta^{2}\ln\theta},$$

$$\pi_{2}\left(\alpha,\lambda,\beta;\theta\right)=(\frac{\lambda}{\alpha\left(1-\theta\right)}\left(-\ln\left(1-\theta\right)\right)^{(1-\alpha)/\alpha}-\frac{\beta}{\theta\ln\theta})^{2}.$$

3.4 Optimum burn-in and replacement time based on failure rate criteria

One application of a bathtub curve is that we can determine the optimum burn-in time in the case when the initial failure rate is too high for the product to be released directly after production. Also, after a certain time, the product enters the wear-out phase and replacement should be considered. The decision can easily be made based on our additive model.

Suppose that the product can only be released after burn-in when the failure rate is less than t_p to meet customers' requirement, then the optimum burn-in time can be determined by

$$\frac{\partial}{\partial t}\int_{0}^{t}H\left(\theta\right)d\theta=t_{p}.$$

From (12), we get that

$$-\ln\left(\frac{\beta}{t\ln t}-\frac{\lambda\left(-\ln\left(1-t\right)\right)^{(1-\alpha)/\alpha}}{\alpha\left(1-t\right)}\right)\simeq t_{p}. \tag{22}$$

The optimum burn-in time should then be the smallest t for which the above equality holds. Because of the shape of the curve, there are usually two solutions to

the above equation and at the point of the first solution, the failure rate has decreased enough to meet the reliability requirement.

Similarly, if the product has to be replaced by a new one when the failure rate is too high, higher than t_c say, then the optimum replacement time can be determined by solving the following equation

$$-\ln\left(\frac{\beta}{t\ln t}-\frac{\lambda\left(-\ln\left(1-t\right)\right)^{(1-\alpha)/\alpha}}{\alpha\left(1-t\right)}\right)\simeq t_c. \tag{23}$$

Unlike the previous case, the largest t for which the above equality holds is the optimum replacement time. After this time, the failure rate is increasing and higher than the acceptable level and a replacement is needed to reduce the risk of immediate failure.

Both (22) and (23) can be solved numerically using standard algorithms.

4 Estimation and Data Analysis

This section is developed to estimate unknown parameters of (7), by its quantile function. The quantile method and L-moments methods are derived and their asymptotic distribution are given.

4.1 The percentile method

For a sample size n, let $T_1,...,T_n$ be independent random variables with common distribution function F, and let $T_{(1)},T_{(2)},...,T_{(n)}$ be ordered lifetimes of n components. Beside the sample distribution function $F_n\left(t\right)=n^{-1}\neq\{k:T_k\leq t,1\leq k\leq n\}$, then the sample percentile function can be represented as

$$\delta_n\left(\theta\right)=\begin{cases} T_{([n\theta]+1)} & \text{if } n\ \theta \text{ is not an integer} \\ T_{([n\theta])} & \text{if } n\ \theta \text{ is an integer.}\end{cases}$$

Nair et al. (2013) have provided asymptotic properties of the percentile estimates. The following theorem studies asymptotic normality of the percentile method.

Theorem 2: [Nair et al. (2013)] Let $T_1,...,T_n$ be independent random variables with common distribution function F, and let $T_{(1)},T_{(2)},...,T_{(n)}$ be ordered lifetimes of n components with quantile function $Q_{\boldsymbol{\Theta}}(\theta)$, density function $f\left(.\right)$ and distribution function $F\left(.\right)$, where $\boldsymbol{\Theta}$ is a vector of k parameters. Assume that $Q_{\boldsymbol{\Theta}}(\theta)$ is unique and $f\left(Q_{\boldsymbol{\Theta}}(\theta)\right)>0$, then $\delta_n\left(\theta\right)\rightarrow Q_{\boldsymbol{\Theta}}(\theta)$ a. s. as $n\rightarrow\infty$ with probability one and $\sqrt{n}\left(\delta_n\left(\theta\right)-Q_{\boldsymbol{\Theta}}(\theta)\right)$ is asymptotically distributed as

$$N\left(0,\frac{\theta\left(1-\theta\right)}{n\left(f\left(Q_{\boldsymbol{\Theta}}(\theta)\right)\right)^2}\right).$$

We equate sample percentile function to population quantile given by

$$\delta_n(\theta) = Q_{\mathbf{\Theta}}(\theta)$$

and

$$\delta_n(1-\theta) = Q_{\mathbf{\Theta}}(1-\theta),$$

for all $\theta < 0.5$.

To illustrate the application of the model (7) we consider the data set represents the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England. They are taken from Smith & Naylor (1987) and for choice $\theta = 0.05$ and $\mu \to 0$: The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. We have the sample percentiles

$$\delta_n(\theta) = 0.81 \text{ and } \delta_n(1-\theta) = 2,$$

and the estimates of the $\mathbf{\Theta} = [\alpha, \beta, \lambda]$ are obtained as

$$\widehat{\alpha} = 1, \quad \widehat{\beta} = -0.675719 \text{ and } \widehat{\lambda} = 1.33758.$$

Plots of failure rate function for different values of θ are given in Figure (3), it depicts $H(\theta)$ is increasing in θ.

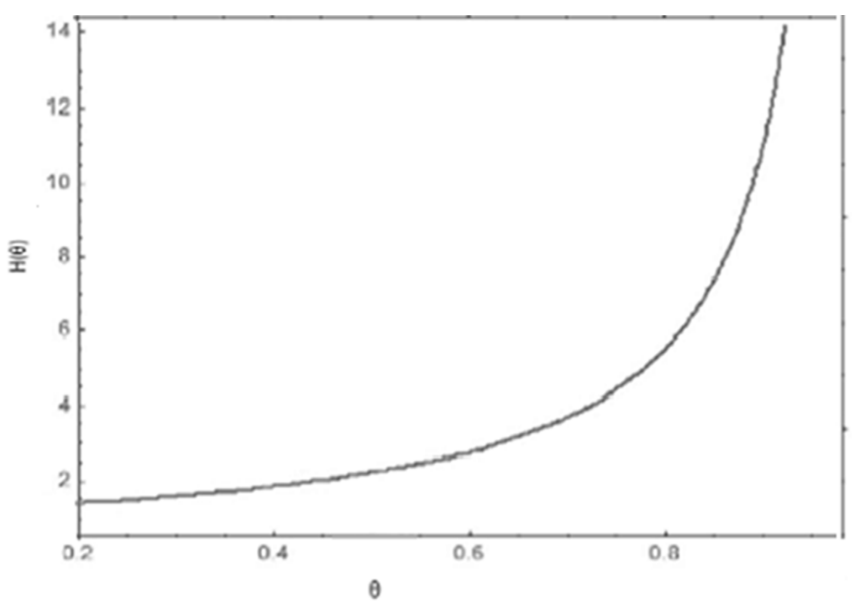

Figure (3): Plots of hazard rate function for different values of θ for real data.

In order to assessing whether (7) model is suited to a data-set, we compare the failure rate function with nonparametric failure rate function with different kernel density functions. We estimate the nonparametric failure rate by

$$\widehat{r}(x) = \frac{\widehat{g}(x)}{\widehat{\overline{G}}(x)},$$

where $\widehat{g}(x)$ is kernel estimator of $g(x)$ and $\widehat{\overline{G}}(x)$ is survival function of $\widehat{f}(x)$. Consider a probability density function $g(u)$ for a lifetime random variable U with distribution function $G(t)$ and survival function $\overline{G}(u) = 1 - G(y)$, $y \in \mathbb{R}^+$.The kernel estimator of $g(x)$ can be computed by the following equation:

$$g(u) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{u - u_i}{h}\right),$$

where h is the bathwidth parameter and K is kernel density. For the real data the automatically computed bandwidth for h by the adaptive method is 0.0453 by domain (0.142, 2.65) and data point 63 and by select Gaussian kernel with mean 0 and standard deviation 1. As can be seen from Figure (4), the nonparametric estimate of failure rate for data set is increasing.

By select some different methods for estimate kernel density such as Gaussian method ($K(u) = \left(1/\sqrt{2\pi}\right)(\exp(-u^2/2)), u \in \mathbb{R}$), Epanechnikov method ($K(u) = \left(3/4\sqrt{5}\right)\left(1 - \frac{u^2}{5}\right), -\sqrt{5} < u < \sqrt{5}$), biweight method ($K(u) = (15/16)(1 - u^2)^2$, $-1 < u < 1$), triangular method ($K(u) = 1 - |u|, -1 < u < 1$), ocsine method ($K(u) = (\pi/4)\cos(\pi u/2), -1 < u < 1$) and semicircle method ($K(u) = (2/\pi)\sqrt{1 - u^2}$,

$-1 < u < 1$). Figure (5) presents these methods.

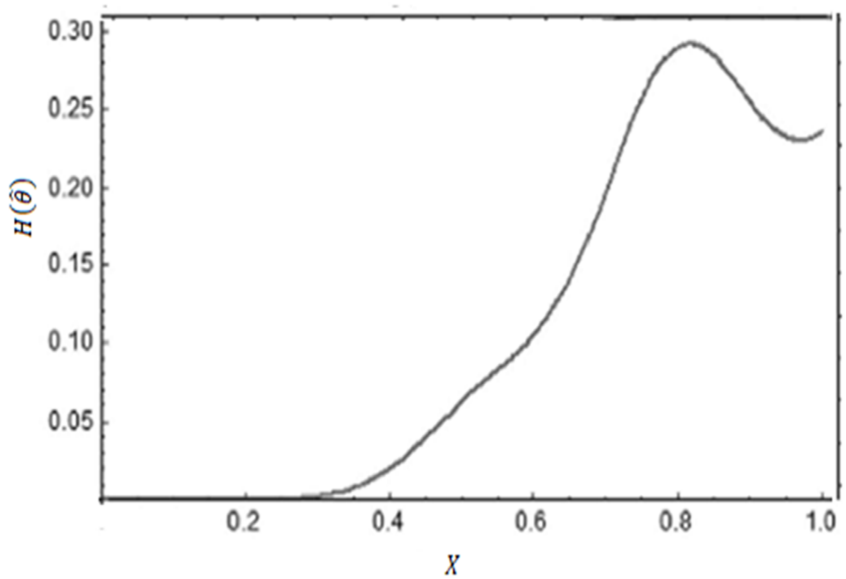

Figure (4): Nonparametric estimate of hazard rate for the data set

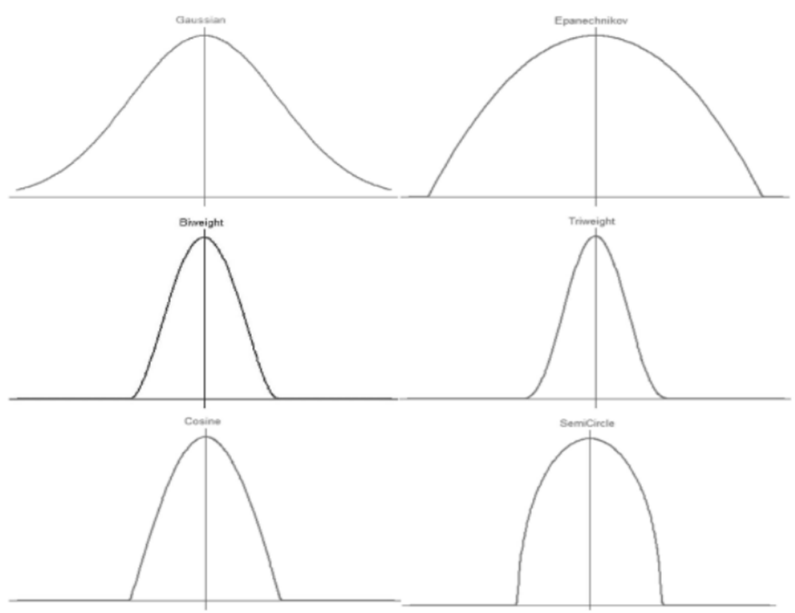

Figure (5): The different kernel functions.

4.2 L-moments method

In the last years, there has been a growing interest in L-moments, because it is a very effective method for an estimate by low bias compared to other estimators. Let

$U_1, ..., U_n$ be independent random variables with common distribution function F and let $U_{(1:n)}, U_{(2:n)}, ..., U_{(n:n)}$ be ordered lifetimes of n components with quantile function $Q_{\boldsymbol{\Theta}}(\theta)$, density function $f(.)$. Since there are four parameters in the model (7), then we consider four L-moments as follows

$$\phi_1 = \binom{n}{1}^{-1} \sum_{s=0}^{n} U_{(s:n)}$$

$$\phi_2 = \frac{1}{2}\binom{n}{2}^{-1} \sum_{s=0}^{n} \left\{ \binom{s-1}{1} - \binom{n-s}{1} \right\} U_{(s:n)}$$

$$\phi_3 = \frac{1}{3}\binom{n}{3}^{-1} \sum_{s=0}^{n} \left\{ \binom{s-1}{2} - 2\binom{n-s}{2}\binom{s-1}{1} + \binom{n-s}{2} \right\} U_{(s:n)}$$

and

$$\phi_4 = \frac{1}{4}\binom{n}{4}^{-1} \sum_{s=0}^{n} \left\{ \begin{array}{c} \binom{s-1}{3} - 3\binom{n-s}{1}\binom{s-1}{2} \\ +3\binom{n-s}{2}\binom{s-1}{1} - \binom{n-s}{3} \end{array} \right\} U_{(s:n)}.$$

The L-moments estimators can be computed by equivalent the population L-moments to sample L-moments as the following equation

$$\phi_r = L_r, \ r = 1, 2, 3, 4. \tag{24}$$

where the population L-moments of the model (7) (L_r) is described by (13). The solution of the set of (24) give the estimates of α, μ, λ and β.

To clarify the procedure of L-moments estimators, we apply the distribution (7) to real data as reported by Efron (1988). The data describe the survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiation therapy. The sample L-moments are $\phi_1 = 226.1738$, $\phi_2 = 94.2896$, $\phi_3 = -65.2683$ and $\phi_4 = -7.2558$. It has been found that $\widehat{\alpha} = 0.389$, $\widehat{\mu} = 200.554$, $\widehat{\lambda} = -47.8531$ and $\widehat{\beta} = 342.493$ by solving the nonlinear set of (24) by using the Newton Raphson method. Figure 6 presents the plot of failure rate quantile function for the data set. From this figure it can be seen that the model has bathtub-shaped failure rate.

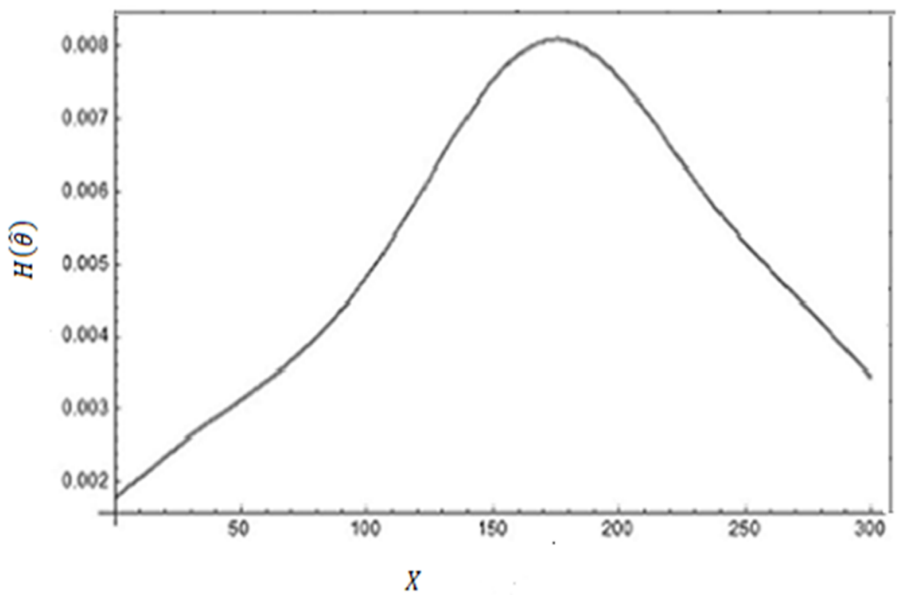

Figure (6): Nonparametric estimate of failure rate for the data set

5 Conclusions

In this article we achieved two goals. The first one is to introduce and study a new family of the failure quantile models with increasing, decreasing and bathtub-shaped failure rates extended distributions and study its statistical and reliability characterizations. The second goal is to estimate the unknown parameters of the new model and provide some applications in the context of statistics and reliability. The obtained results are validated using two real data sets and it is shown that the new family provides a better fit than some other known distributions.

On the basis of the promising findings presented in this paper, work on the remaining issues is continuing and will be presented in future papers such as

(i) Discuss the mixture analysis of the new family.

(ii) Introduce and study bivariate distributions of failure quantile models.

(iii) Provide the bayesian estimation with different lose functions of proposed model.

References

[1] Ahmad, I.A., and Kayid, M. (2005). Characterizations of the RHR and MIT ordering and the $DRHR$ and $IMIT$ classes of life distributions. *Probability in the Engineering and Informational Sciences,* 19, 447–461.

[2] Ahmad, I.A., and Kayid, M. (2006). Characterization results of the $NBUCA$ classes of life distributions. *International Journal of Reliability & Applications*, 7, 1, 79-87.

[3] Ahmad, I.A., and Kayid, M. (2007). Reversed preservation of stochastic orders for random minima and maxima with applications. *Statistical Papers*, 48, 283-293.

[4] Bassett, G., and Koenker, R. (1982). An empirical quantile function for linear models with i.i.d. errors. *Journal of the American Statistical Association*, 77, 407–415.

[5] Castillo, E., and Hadi, A.S. (1994). Parameter and quantile estimation for the generalized extreme-value distribution. *Econometrics*, 5, 417–432.

[6] Efron, B. (1988). Logistic regression survival analysis and the Kaplan-Meier curve. Journal of the American Statistical Association 83(402): 414-425

[7] Fuller, W.E. (1914). Flood flows. *Transactions of the American Society of Civil Engineers*, 77, 564.

[8] Gilchrist, W. (2000). Statistical Modelling with Quantile Functions. CRC Press, Abingdon.

[9] Gumbel, E.J. (1935). Les valeurs extrêmes des distributions statistiques. *Annales de l'Institut Henri Poincaré*, 5, 2, 115–158

[10] Gumbel, E.J. (1941). The return period of flood flows. *The Annals of Mathematical Statistics*, 12, 163–190.

[11] Griffith, A.A. (1920). The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society of London*, Series A, 221, 163-198.

[12] Haupt, E., and Schabe, H. (1992). A new model for a lifetime distribution with bathtub shaped failure rate. *Microelect. & Reliab.*, 32 (1992) 633-639.

[13] Kupka, J., and Loo, S. (1989). The hazard and vitality measures of aging. *Journal of Applied Probability*, 26, 3, 532-542.

[14] Kus, C. (2007). A new lifetime distribution. Comput. Statist. Data Anal. 51 (9), 4497–4509.

[15] Li, X., and Zuo, M.J. (2004). Preservation of stochastic orders for random minima and maxima with applications. *Naval Research Logistics*, 51, 332-344.

[16] Li, Y. (2004). Closure of $NBU(2)$ class under the formation of parallel systems. *Statistics and Probability Letters*, 67, 57-63.

[17] Lomax, K.S. (1954). Business failure: Another example of teh analysis of failure data. J. Amer. Statist. Assoc. 49, 847–852.

[18] Muldholkar, G.S., and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Trans. Reliab., 11-42, 299-303.

[19] Nair, N.U., and Sankaran, P.G. (2009). Quantile-Based Reliability Analysis. *Communications in Statistics - Theory and Methods,* 38, 2, 222-232.

[20] Nair, N.U, and Vineshkumar, B. (2010). L-moments of residual life. *Journal of Statistical Planning and Inference,* 140, 9, 2618-2631.

[21] Nair, N.U., Sankaran, P.G., Kumar, B.V. (2012). Modelling lifetimes by quantile functions using parzen's score function. *Statistics,* 46, 6, 799–811.

[22] Nair, N.U., Sankaran, P.G., and Balakrishnan, N. (2013). Quantile-Based Reliability Analysis. Birkhäuser, New York, NY.

[23] Nassar, M.M. (1988). Two properties of mixtures of exponential distributions. *IEEE Trans. Reliab.* 37 (4), 383–385.

[24] Noughabi, M. S., and Izadkhah, S. (2016). On the quantile past lifetime of the components of the parallel systems. *Communications in Statistics - Theory and Methods*, 45, 7, 2130-2136 .

[25] Parzen, E. (1979). Non parametric statistical data modelling, *Journal of the American Statistical Association,* 74, 105–122

[26] Sankaran, P.G., Thomas, B., and Midhu, N.N. (2015). On bilinear hazard quantile functions. *Metron*, 73, 1, 135–148.

[27] Sankaran, P.G., Nair, N. U., and Midhu, N.N. (2016). A new quantile function with applications to reliability analysis. *Communications in Statistics-Simulation and Computation*, 45, 2, 566–582.

[28] Sankaran, P.G., and Kumar, D. (2018). New class of quantile functions useful in reliability analysis. *Journal of Statistical Theory and Practice*, accepted.

[29] Smith, R. L., and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Statistics 36, 358-369

[30] Sunoj, S.M., Sankaran, P.G., and Nair, N.U. (2017). Quantile-Based Reliability Aspects of Rényi's Information Divergence Measure. *Journal of the Indian Society for Probability and Statistics*, 18, 2, 267–280.